Problem Sheet 4

1. a) Evaluate
$$\sum_{n\le x}\frac{\sigma(n)}{n}.$$
b) By Partial Summation deduce
$$\sum_{n\le x}\sigma(n)=\frac{\zeta(2)}{2}x^2+O(x\log x)\,.$$

Hint Find f such that $\sigma(n)/n=\sum_{d|n}f(d)$.

2. On Problem Sheet 3 it was shown that $Q_k=1*\mu_k$ where Q_k is the characteristic function of the k-free integers while $\mu_k(a)=1$ if $a=m^k$ for some integer m, 0 otherwise. Use the Convolution Method to show that
$$\sum_{n\le x}Q_k(n)=\frac{x}{\zeta(k)}+O\big(x^{1/k}\big)\,,$$
for $k\ge 2$.

3. Look back in the notes to recall how the result
$$\sum_{n\le x}\frac{1}{n}=\log x+O(1)$$
was improved to
$$\sum_{n\le x}\frac{1}{n}=\log x+\gamma+O\left(\frac{1}{x}\right). \qquad (13)$$
Use the same method to improve Corollary 4.7 to
$$\sum_{n\le x}\frac{Q_2(n)}{n}=\frac{1}{\zeta(2)}\log x+C+O\left(\frac{1}{x^{1/2}}\right),$$
for some constant C.

(Unfortunately, this doesn't directly lead to an improvement for $\sum_{n\le x}2^{\omega(n)}$, just as (13) doesn't directly lead to an improvement for $\sum_{n\le x}d(n)$.)

4. Prove by induction that

$$\sum_{n\le x} d_\ell(n) = \frac{1}{(\ell-1)!} x \log^{\ell-1} x + O\left(x \log^{\ell-2} x\right) \qquad (14)$$

for all $\ell \ge 2$.

Hint Assuming (14) use partial summation to prove a result for $\sum_{n\le x} d_\ell(n)/n$. Then use $d_{\ell+1} = 1 * d_\ell$ to get a result for $\sum_{n\le x} d_{\ell+1}(n)$.

5. a) Prove that

$$\sum_{n\le x} \frac{\sigma(n)}{n^2} = \zeta(2)\log x + O(1)\,.$$

b) Prove that

$$\log x \ll \sum_{n\le x} \frac{1}{\phi(n)} \ll \log x.$$

Hint for part b) use a result from Problem Sheet 3 which combines $\sigma(n)$ and ϕ.

6. In Problem Sheet 4 the characteristic function, q_2, of square-full numbers was defined. So on prime powers $q(p^a) = 0$ if $a = 1$, 1 if $a \ge 2$. It was shown there that

$$\sum_{n=1}^{\infty} \frac{q_2(n)}{n^s} = \zeta(2s)\frac{\zeta(3s)}{\zeta(6s)}. \qquad (15)$$

Recall that

$$\sum_{n=1}^{\infty} \frac{Q_2(n)}{n^s} = \frac{\zeta(s)}{\zeta(2s)} \qquad (16)$$

Therefore

$$\begin{aligned}\frac{\zeta(3s)}{\zeta(6s)} &= \frac{\zeta(3s)}{\zeta(2(3s))} = \sum_{m=1}^{\infty} \frac{Q_2(m)}{m^{3s}} \\ &= \sum_{m=1}^{\infty} \frac{Q_2(m)}{(m^3)^s} = \sum_{\substack{n=1\\ n=m^3}}^{\infty} \frac{Q_2(m)}{n^s} = \sum_{n=1}^{\infty} \frac{h(n)}{n^3},\end{aligned}$$

where

$$h(n) = \begin{cases} Q_2(m) & \text{if } n = m^3 \\ 0 & \text{otherwise.} \end{cases}$$

Thus (15) can be written as

$$D_{q_2}(s) = \zeta(2s)\,\frac{\zeta(3s)}{\zeta(6s)} = D_{sq}(s)\,D_h(s) = D_{sq*h}(s)\,,$$

where $sq\,(n) = 1$ if n is a square, 0 otherwise. This suggests that $q_2 = sq * h$

i) Prove that

$$q_2 = sq * h,$$

by showing that the two sides agree on all prime powers.

ii) Use the Composition Method to prove

$$\sum_{n\le x} q_2(n) = x^{1/2}\frac{\zeta(3/2)}{\zeta(3)} + O\big(x^{1/3}\big)\,.$$

Note

$$\sum_{n\le x} sq(x) = x^{1/2} + O(1)\,,$$

so

$$\sum_{n\le x} q_2(n) \sim \frac{\zeta(3/2)}{\zeta(3)} \sum_{n\le x} sq(x)\,.$$

The coefficient here is approximately 2.173254...., so we might say that there are just over twice as many square-full integers as squares

7. If $f = 1 * g$ the convolution method starts with

$$\sum_{1\le n\le x} f(n) = \sum_{1\le a\le x} g(a)\left[\frac{x}{a}\right]. \tag{17}$$

Use this equality to show that for Euler's phi function ϕ and integral N we have

$$\sum_{1\le a\le N} \left[\frac{N}{a}\right]\phi(a) = \frac{1}{2}N\,(N{+}1)\,.$$

8. i. Recall
$$\sum_{n \le x} d(n) = x \log x + O(x)\,.$$
Prove, by using Partial Summation on this result, that
$$\sum_{n \le x} nd(n) = \frac{1}{2}x^2 \log x + O\big(x^2\big)\,.$$

ii. In a previous Question Sheet you showed that $\sigma * \phi = j * j$. Use this to show that
$$\sum_{n \le x} (\sigma * \phi)(n) = \frac{1}{2}x^2 \log x + O\big(x^2\big)\,.$$

Hint make use of part i.